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LETTER TO THE EDITOR

Shape invariant potentials with $\mathcal{PT}$ symmetry

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Abstract. Suitable complexification of the well known hyperbolic Rosen–Morse oscillator in one dimension is shown to give the fourth (and last?) exactly solvable model which combines the shape and $\mathcal{PT}$ invariance.

Review paper [1] emphasizes that a requirement of the so-called shape invariance (with respect to the translation of a parameter) determines all the four known exactly solvable analytic potentials in one dimension:

- the shifted harmonic oscillator $V^{[H]}(x) = (\mu x - b)^2$,
- the Morse potential $V^{[M]}(x) = A^2 + b^2 \exp(-2\mu x) - (2A + \mu)b \exp(-\mu x)$,
- the 'scarf' $V^{[S]}(x) = [B^2 - A^2 - \mu A + (2A + \mu)B \sinh \mu x]/ \cosh^2 \mu x$, and
- the Rosen–Morse (RM) well $V^{[\mathrm{RM}]}(x) = -A(A + \mu)/ \cosh^2 \mu x + 2C \sinh \mu x/ \cosh \mu x$ with $C < A^2$.

An increasingly complicated parameter dependence of the respective spectra

$$\begin{array}{ll} E_n^{[H]} = \mu n + \mu \qquad E_n^{[M]} = E_n^{[S]} = -(A - \mu n)^2 \\ E_n^{[\mathrm{RM}]} = -(A - \mu n)^2 - C^2/(A - \mu n)^2 \end{array} \tag{1}$$

contributes to the popularity of the harmonic $V^{[H]}$. The most recent illustration of this not quite deserved preference is offered by the $\mathcal{PT}$-symmetric quantum mechanics [2–6]. One of its first presentations by Bender and Boettcher [2] started from the illustrative $V^{[H]}(x)$. This potential becomes $\mathcal{PT}$-symmetric (which, in essence, means that $V(-x) = [V(x)]^*$) at the purely imaginary parameters b. The $\mathcal{PT}$-symmetric version of the next model $V^{[M]}(x)$ is more complicated but it is already also understood in fair detail [7]. In contrast, only the first few remarks on $V^{[S]}(x)$ did just appear in the letter [8], presenting its $\mathcal{PT}$-symmetrized form as a supersymmetric partner of its real (and spatially symmetric, also known as the modified or hyperbolic Pöschl–Teller [9]) special case. Up to now, no news seem available about $V^{[\mathrm{RM}]}(x)$. Indeed, the trick of [8] does not apply as the spectrum would change with the naive $\mathcal{PT}$-symmetrization $C = \mathrm{i}c$ where $c =$ real. The gap is to be filled by this letter.

As a preparatory step let us recollect that without any supersymmetric considerations [5] the manifestly $\mathcal{PT}$-symmetric version of the 'scarf' potentials is obtained at $B = \mathrm{i}b$ with real $b \neq 0$. Without any loss of generality we may fix the scale $\mu = 1$ and take the well known Jacobi-polynomial wavefunction formulae, e.g., from table 4.1 of [1]. *Mutatis mutandis* we get

$$\psi_n^{[S]}(x) = \frac{\mathrm{i}^n}{\cosh^A x} \mathrm{e}^{-\mathrm{i}b \arctan(\sinh x)}\ P_n^{(-b-A-1/2,b-A-1/2)}(\mathrm{i} \sinh x).$$

These $\mathcal{PT}$-symmetric solutions remain well behaved along the whole real line. They vanish asymptotically and remain normalizable for all the non-negative integers $n < A$. The sample solutions of [8] reappear here as special cases.

A step towards the new RM-type model

$$V^{[\mathrm{RM}]}(x) = -\frac{A(A+1)}{\cosh^2 x} + 2\mathrm{i}c\frac{\sinh x}{\cosh x} \tag{2}$$

must be made with more care. Indeed, the energy formula (1) changes its meaning since we have to insert the negative $C^2 = -c^2 < 0$. At the same time the analytically continued wavefunctions of [1] preserve their explicit and transparent form

$$\psi_n^{[\mathrm{RM}]}(x) = \frac{1}{\cosh^{A-n} x}\mathrm{e}^{-\mathrm{i}cx/(A-n)}\ P_n^{(A-n+\mathrm{i}c/(A-n),A-n-\mathrm{i}c/(A-n))}(\tanh x).$$

They exhibit an asymptotic decrease for all the same non-negative integers $n < A$ as above. Nevertheless, due to the asymmetry $V^{[\mathrm{RM}]}(\pm\infty) = \pm 2\mathrm{i}c$ we recover an apparent paradox. In our RM potential which is asymptotically purely imaginary, bound states are formed even at positive energies. Indeed, we have $E_n^{[\mathrm{RM}]} > 0$ at all the integers $n \in [n_{\mathrm{crit}}, n_{\mathrm{max}}]$ such that $n_{\mathrm{crit}} > A - \sqrt{c} \geqslant 0$ (with, of course, $n_{\mathrm{max}} < A$). The phenomenon represents an exactly solvable parallel to the empirically observed positivity of energies in the $\mathcal{PT}$-symmetric model $V^{[\mathrm{ZJB}]}(x) = -\mathrm{i}x^3$ of Zinn-Justin and Bessis which is asymptotically purely imaginary as well [3, 10].

A transition to the more general $\mathcal{PT}$-symmetric power law forces $V^{[\mathrm{BB}]}(x) = (-\mathrm{i}x)^\delta$ of Bender and Boettcher [2] enables us to draw still one more parallel. Indeed, the (numerical and semiclassical) analysis indicates a decay of the high-lying bound states $E_n^{[\mathrm{BB}]} = E_n^{[\mathrm{BB}]}(\delta)$ for decreasing $\delta < 2$. Below a certain $\delta_{\mathrm{crit}} < 2$ one is left with the mere single (i.e., ground) bound state. This is the last real level which disappears finally in Herbst's limit: $E_n^{[\mathrm{BB}]}(\delta) \to +\infty$ for $\delta \to 1^+$ [11]. In this context, the possible connection between the asymptotic growth of $|V(x)|$ and a decay of the high-lying bound states is given a different form in our $|V^{[\mathrm{RM}]}(\pm\infty)| < \infty$ example. Even at a medium imaginary part (and a really small real component) our model (2) still supports a single high-lying ground state at a (variably and arbitrarily) large energy $E_0^{[\mathrm{RM}]} = c^2/A^2 + \mathcal{O}(A^2)$.

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